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Cho-Duan-Ge decomposition of QCD in the constraintless Clairaut-type formalism

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We apply the recently derived constraintless Clairaut-type formalism to the Cho-Duan-Ge decomposition in $SU(2)$ QCD. We find nontrivial corrections to the physical equations of motion and that the contribution of the topological degrees of freedom is qualitatively different from that found by treating the monopole potential as though it were dynamic. We also find alterations to the field commutation relations that undermine the particle interpretation in the presence of the chromomonopole condensate.

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I. INTRODUCTION

The occurrence of "redundant" degrees of freedom not determined by equations of motion (EOMs) is a characteristic property of any physical system having symmetry [1,2]. In gauge theories, the covariance of EOMs under symmetry transformations leads to gauge ambiguity, i.e., the appearance of undetermined functions. In this situation, some dynamical variables obey first order differential equations [3]. One then employs a suitably modified Hamiltonian formalism, such as the Dirac theory of constraints [4].

A constraintless generalization of the Hamiltonian formalism based on a Clairaut-type formulation was recently put forward by one of the authors [5,6]. It generalizes the standard Hamiltonian formalism to include Hessians with zero determinant, providing a rigorous treatment of the nonphysical degrees of freedom in the derivation of EOMs and the quantum commutation relations. An outline is given in the Appendix.

The Cho-Duan-Ge (CDG) decomposition of the gluon field in quantum chromodynamics (QCD) published by Duan and Ge [7] and also by Cho [8] specifies the Abelian components of the background field in a gauge covariant manner. In so doing, it identifies the monopole degrees of freedom (d.o.f.) of the gluon field naturally, making it preferable to the conventional maximal Abelian gauge [9]. It can also generate a gauge invariant canonical momentum, which makes it of interest to studies of nucleon spin decomposition [10–13].

Up until now, the monopole d.o.f. have not been rigorously handled. Indeed, merely accounting for the physical and gauge d.o.f. proved to be a long and difficult task [14–18]. An important observation of the monopole d.o.f. by Cho *et al.* is that the Euler-Lagrange equation for the Abelian direction does not yield a new EOM. Their interpretation is that the monopole is the "slow-changing background part" of the gauge field while the physical gluons constituted the "fast-changing quantum part."

In this paper, we apply the Clairaut formalism to the monopole d.o.f. in two-color QCD. We consider both the gluon field and scalar "quarks" in the fundamental field. We find that the interaction between monopole and physical d.o.f. vanishes from the EOMs but that the canonical commutation relations are altered in a manner that leaves the particle number undefined.

Section II describes the CDG decomposition and establishes notation. In Sec. III, we identify the field theory equivalent of q^α and go on to find the q^α curvature in Sec. IV. The curvature's nonzero value leads to alterations in the EOMs elucidated in Sec. V, while corresponding results are found in Sec. VI for color-charged scalars in the fundamental representation. Our most important results, alterations to the commutation relations and their implications for the particle interpretation, are discussed in Sec. VII. We give a final discussion in Sec. VIII and a detailed summary of the Clairaut formalism in the Appendix.

II. REPRESENTING THE GLUON FIELD

The CDG decomposition [7,8] and another like it [19] was (re)discovered [14] at about the turn of the century when several groups were readdressing the stability of the chromomonopole condensate [15–17,20–22]. Some authors [16,17,21], including one of the current ones [22], have overlooked the differences between the CDG decomposition and that of Faddeev and Niemi, referring to the former as either the Cho-Faddeev-Niemi or the Cho-Faddeev-Niemi-Shabanov decomposition. In this paper, we label it the CDG decomposition, as per the convention of Cho *et al.* [13].

The Lie group $SU(N)$ has N^2-1 generators $\lambda^{(a)}$ ($a=1,\ldots,N^2-1$), of which $N-1$ are Abelian generators

$\Lambda^{(i)}$ $(i = 1, \ldots, N-1)$. The gauge transformed Abelian directions (Cartan generators) are denoted as

$$\hat{n}_i(x) = U(x)^\dagger \Lambda^{(i)} U(x). \tag{1}$$

Gluon fluctuations in the $\hat{n}_i(x)$ directions are described by $c_\mu^{(i)}(x)$, where μ is the Minkowski index. There is a covariant derivative which leaves the $\hat{n}_i(x)$ invariant,

$$\hat{D}_\mu \hat{n}_i(x) \equiv (\partial_\mu + g\vec{V}_\mu(x)\times)\hat{n}_i(x) = 0, \tag{2}$$

where $\vec{V}_\mu(x)$ is of the form

$$\begin{aligned}\vec{V}_\mu(x) &= c_\mu^{(i)}(x)\hat{n}_i(x) + \vec{C}_\mu(x),\\ \vec{C}_\mu(x) &= g^{-1}\partial_\mu \hat{n}_i(x) \times \hat{n}_i(x).\end{aligned} \tag{3}$$

The vector notation refers to the internal space, and summation is implied over $i = 1, \ldots, N-1$. For later convenience, we define

$$F_{\mu\nu}^{(i)}(x) = \partial_\mu c_\nu^{(i)}(x) - \partial_\nu c_\mu^{(i)}(x), \tag{4}$$

$$\begin{aligned}\vec{H}_{\mu\nu}(x) &= \partial_\mu \vec{C}_\nu(x) - \partial_\nu \vec{C}_\mu(x) + g\vec{C}_\mu(x) \times \vec{C}_\nu(x)\\ &= H_{\mu\nu}^{(i)}(x)\hat{n}_i(x),\end{aligned} \tag{5}$$

$$H_{\mu\nu}^{(i)}(x) = \vec{H}_{\mu\nu}(x) \cdot \hat{n}_i(x). \tag{6}$$

The vectors $\vec{X}_\mu(x)$ denote the dynamical components of the gluon field in the off-diagonal directions of the internal space, so if $\vec{A}_\mu(x)$ is the gluon field, then

$$\vec{A}_\mu(x) = \vec{V}_\mu(x) + \vec{X}_\mu(x) = c_\mu^{(i)}(x)\hat{n}_i(x) + \vec{C}_\mu(x) + \vec{X}_\mu(x), \tag{7}$$

where

$$\vec{X}_\mu(x) \perp \hat{n}_i(x), \quad \forall\ 1 \le i < N, \qquad \vec{D}_\mu = \partial_\mu + g\vec{A}_\mu(x). \tag{8}$$

The Lagrangian density is still

$$\mathcal{L}_{\text{gauge}}(x) = -\frac{1}{4}\vec{F}_{\mu\nu}(x) \cdot \vec{F}^{\mu\nu}(x), \tag{9}$$

where the field strength tensor of QCD expressed in terms of the CDG decomposition is

$$\begin{aligned}\vec{F}_{\mu\nu}(x) = (F_{\mu\nu}^{(i)}(x) + H_{\mu\nu}^{(i)}(x))\hat{n}_i(x) + (\hat{D}_\mu \vec{X}_\nu(x)\\ - \hat{D}_\nu \vec{X}_\mu(x)) + g\vec{X}_\mu(x) \times \vec{X}_\nu(x).\end{aligned} \tag{10}$$

We will later have a need for the conjugate momenta. These are only defined up to a gauge transformation, so to avoid complications, we take the Lorenz gauge. The conjugate momentum for the Abelian component is then

$$\Pi^{(i)\mu}(x) = \frac{\delta(\int d^3x \mathcal{L}_{\text{gauge}})}{\delta \partial_0 c_\mu^{(i)}(x)} = -\vec{F}^{0\mu}(x) \cdot \hat{n}^{(i)}(x), \tag{11}$$

while the conjugate momentum of $\vec{X}_\mu(x)$ is

$$\begin{aligned}\vec{\Pi}^\mu(x) &= \frac{\delta(\int d^3x \mathcal{L}_{\text{gauge}})}{\delta \hat{D}_0 \vec{X}_\mu(x)}\\ &= -\frac{1}{2}(\hat{D}^0 \vec{X}^\mu(x) - \hat{D}^\mu \vec{X}^0(x)\\ &\quad + g(\vec{X}^\mu(x) \times \vec{X}^\nu(x))_{\perp\{\hat{n}^{(i)}:1\le i\le M\}}).\end{aligned} \tag{12}$$

From now on, we restrict ourselves to the $SU(2)$ theory for which there is only one $\hat{n}(x)$ lying in a three-dimensional internal space and neglect the (i) indices. The results can be extended to larger $SU(N = M+1)$ gauge groups [22], although the cross product in Eq. (12) vanishes when $N = 2$.

The above outline neglects various mathematical subtleties involved in a fully consistent application of the CDG decomposition. In fact, its proper interpretation and gauge fixing took considerable effort by several independent groups. The interested reader is referred to [14–18] for further details.

III. THE q^α GAUGE FIELDS OF THE MONOPOLE FIELD

Now we adapt the Clairaut approach (see the Appendix) [5,23] to quantum field theory and apply it to the CDG decomposition of the QCD gauge field, leaving the fundamental representation until Sec. VI. Substituting the polar angles

$$\begin{aligned}\hat{n}(x) = \cos\theta(x)\sin\phi(x)\hat{e}_1 + \sin\theta(x)\sin\phi(x)\hat{e}_2\\ + \cos\phi(x)\hat{e}_3\end{aligned} \tag{13}$$

and defining

$$\begin{aligned}\sin\phi(x)\hat{n}_\theta(x) &\equiv \int dy^4 \frac{d\hat{n}(x)}{d\theta(y)} = \sin\phi(x)(-\sin\theta(x)\hat{e}_1\\ &\quad + \cos\theta(x)\hat{e}_2),\\ \hat{n}_\phi(x) &\equiv \int dy^4 \frac{d\hat{n}(x)}{d\phi(y)} = \cos\theta(x)\cos\phi(x)\hat{e}_1\\ &\quad + \sin\theta(x)\cos\phi(x)\hat{e}_2 - \sin\phi(x)\hat{e}_3\end{aligned} \tag{14}$$

for later convenience, we note that

$$\hat{n}_{\phi\phi} = -\hat{n},$$
$$\sin\phi(x)\hat{n}_{\theta\theta} = -\sin\phi(x)(\cos\theta\hat{e}_1 + \sin\theta\hat{e}_2), \quad (15)$$

and that the vectors $\hat{n}(x) = \hat{n}_\phi(x)\times\hat{n}_\theta(x)$ form an orthonormal basis of the internal space.

Substituting the above into the Cho connection in Eq. (3) gives

$$\begin{aligned} g\vec{C}_\mu(x) &= (\cos\theta(x)\cos\phi(x)\sin\phi(x)\partial_\mu\theta(x) \\ &\quad + \sin\theta(x)\partial\phi(x))\hat{e}_1 \\ &\quad + (\sin\theta(x)\cos\phi(x)\sin\phi(x)\partial_\mu\theta(x) \\ &\quad - \cos\theta(x)\partial\phi(x))\hat{e}_2 - \sin^2\phi(x)\partial_\mu\theta(x)\hat{e}_3 \\ &= \sin\phi(x)\partial_\mu\theta(x)\hat{n}_\phi(x) - \partial_\mu\phi(x)\hat{n}_\theta(x) \end{aligned} \quad (16)$$

from which it follows that

$$\begin{aligned} g^2\vec{C}_\mu(x)\times\vec{C}_\nu(x) &= \sin\phi(x)(\partial_\mu\phi(x)\partial_\nu\theta(x) \\ &\quad - \partial_\nu\phi(x)\partial_\mu\theta(x))\hat{n}(x). \end{aligned} \quad (17)$$

Treating θ, ϕ as dynamic variables, their conjugate momenta are

$$\begin{aligned} \bar{p}_\phi(x) &= \int dy^3 \frac{\delta\mathcal{L}}{{}_x\partial_0\phi(x)} \\ &= \int dy^3 \int dy^0 \delta(x^0 - y^0)(\sin\phi(y)_y\partial^\mu\theta(y)\hat{n}(y) \\ &\quad + \hat{n}_\theta(y)\times\vec{X}^\mu(y))\cdot\vec{F}_{0\mu}(y)\delta^3(\vec{x}-\vec{y}) \\ &= (\sin\phi(x)\partial^\mu\theta(x)\hat{n}(x) + \hat{n}_\theta(x)\times\vec{X}^\mu(x))\cdot\vec{F}_{0\mu}(x), \end{aligned} \quad (18)$$

$$\begin{aligned} \bar{p}_\theta(x) &= \int dy^3 \frac{\delta\mathcal{L}}{{}_x\partial_0\theta(x)} \\ &= -\int dy^3 \int dy^0 \delta(x^0 - y^0)\sin\phi(y)({}_y\partial^\mu\phi(y)\hat{n}(y) \\ &\quad + \sin\phi(y)\hat{n}_\phi(y)\times\vec{X}^\mu(y))\cdot\vec{F}_{0\mu}(y)\delta^3(\vec{x}-\vec{y}) \\ &= -\sin\phi(x)(\partial^\mu\phi(x)\hat{n}(x) + \hat{n}_\phi(x)\times\vec{X}^\mu(x))\cdot\vec{F}_{0\mu}(x). \end{aligned} \quad (19)$$

The Hessian is given by

$$\|\frac{\delta^2\mathcal{L}}{\delta q^A\delta q^B}\| = 0, \quad (20)$$

where A, B run over all fields, both physical and topological. It follows from inspection of the Lagrangian density, Eqs. (9) and (10), that the time derivatives of $\theta(x)$, $\phi(x)$ occur only in linear combination with those of one of the physical gluon fields $c_\mu(x)$, $\vec{X}_\mu(x)$, either through $F_{0\nu}(x) + H_{0\nu}(x)$ or $\hat{D}_0$. (This is readily extended to quarks, which we introduce in Sec. VI). Therefore, the rows (columns) of the Hessian matrix corresponding to $\dot{\theta}(x)$, $\dot{\phi}(x)$ must be linear combinations of those corresponding to the physical field velocities, so the Hessian vanishes.

This linear dependence within the Hessian is consistent with Cho and Pak's [15] and Bae *et al.*'s [18] finding that $\hat{n}(x)$ (and by extension $\theta(x)$, $\phi(x)$) does not generate an independent EOM.

We, therefore, use the discussion surrounding (3.10) in [5] and define

$$B_\theta(x) \equiv \bar{p}_\theta(x), \qquad B_\phi(x) \equiv \bar{p}_\phi(x), \quad (21)$$

where the definitions of $B_\phi(x)$, $B_\theta(x)$ are generalized to quantum field theory from those in [5]. It follows that $H_{\rm phys} = H_{\rm mix}$ (also defined in [5]).

IV. THE q^α CURVATURE

From Eqs. (18) and (19), we have

$$\frac{\delta B_\phi(x)}{\delta\theta(y)} = (\sin\phi(x)\hat{n}_{\theta\theta}(x)\times\vec{X}^\mu\cdot\vec{F}_{0\mu}(x) - T_\phi(x))\delta^4(x-y), \quad (22)$$

$$\begin{aligned} \frac{\delta B_\theta(x)}{\delta\phi(y)} &= -(\cos\phi(x)(\partial^\mu\phi(x)\hat{n}(x) + \hat{n}_\phi(x)\times\vec{X}^\mu(x)) \\ &\quad \cdot(\vec{F}_{0\mu}(x) + \vec{H}_{0\mu}) + T_\theta(x))\delta^4(x-y), \end{aligned} \quad (23)$$

where

$$\begin{aligned} T_\phi(x) &= \partial^k[\sin\phi(x)\hat{n}\cdot\vec{F}_{0k}(x) - (\sin\phi(x)\partial_k\theta(x) \\ &\quad + \hat{n}_\theta(x)\times\vec{X}_k\cdot\hat{n})\partial_0\phi(x)], \end{aligned} \quad (24)$$

$$\begin{aligned} T_\theta(x) &= -\partial^k[\sin\phi(x)(\hat{n}\cdot\vec{F}_{0k}(x) + (\partial_k\phi(x) \\ &\quad + \hat{n}_\phi(x)\times\vec{X}_k\cdot\hat{n})\partial_0\theta(x))] \end{aligned} \quad (25)$$

are the surface terms arising from derivatives $\frac{\delta(\partial\theta)}{\delta\theta}$, $\frac{\delta(\partial\phi)}{\delta\phi}$, and the latin index k is used to indicate that only spatial indices are summed over.

This yields the q^α curvature

$$\begin{aligned} \mathcal{F}_{\theta\phi}(x) &= \int dy^4\left(\frac{\delta B_\theta(x)}{\delta\phi(y)} - \frac{\delta B_\phi(x)}{\delta\theta(y)}\right)\delta^4(x-y) \\ &\quad + \{B_\phi(x), B_\theta(x)\}_{\rm phys} \\ &= -\cos\phi(x)(\partial^\mu\phi(x)\hat{n}(x) + \hat{n}_\phi(x) \\ &\quad \times\vec{X}^\mu(x))\cdot(\vec{F}_{0\mu}(x) + \vec{H}_{0\mu}(x)) \\ &\quad - \sin\phi(x)\hat{n}_{\theta\theta}(x)\times\vec{X}^\mu(x)\cdot\vec{F}_{\mu0}(x) \\ &\quad + T_\phi(x) - T_\theta(x), \end{aligned} \quad (26)$$

where we have used that the bracket $\{B_\phi(x), B_\theta(x)\}_{\mathrm{phys}}$ vanishes because $B_\phi(x)$ and $B_\theta(x)$ share the same dependence on the dynamic d.o.f. and their derivatives.

In earlier work on the Clairaut formalism [5,23], this was called the q^α-field strength, but we call it q^α curvature in quantum field theory applications to avoid confusion.

This nonzero $\mathcal{F}^{\theta\phi}(x)$ is necessary and usually sufficient to indicate a nondynamic contribution to the conventional Euler-Lagrange EOMs. More significant is a corresponding alteration of the quantum commutators, with repercussions for canonical quantization and the particle number.

V. ALTERED EQUATIONS OF MOTION

Generalizing Eqs. (7.1), (7.3), and (7.5) in [5]

$$\begin{aligned}\partial_0 q(x) &= \{q(x), H_{\mathrm{phys}}\}_{\mathrm{new}} \\ &= \frac{\delta H_{\mathrm{phys}}}{\delta p(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta p(x)} \partial^0 \alpha(y),\end{aligned} \tag{27}$$

the derivative of the Abelian component complete with corrections from the monopole background is

$$\partial_0 c_\sigma(x) = \frac{\delta H_{\mathrm{phys}}}{\delta \Pi^\sigma(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta \Pi^\sigma(x)} \partial^0 \alpha(y). \tag{28}$$

The effect of the second term is to remove the monopole contribution to $\frac{\delta H_{\mathrm{phys}}}{\delta \Pi^\sigma(x)}$. To see this, consider that, by construction, the monopole contribution to the Lagrangian and Hamiltonian is dependent on the time derivatives of θ, ϕ, so the monopole component of $\frac{\delta H_{\mathrm{phys}}}{\delta \Pi^\sigma(x)}$ is

$$\begin{aligned}&\frac{\delta}{\delta \Pi^\sigma(x)} H_{\mathrm{phys}}|_{\dot\theta \dot\phi} \\ &\quad= \frac{\delta}{\delta \Pi^\sigma(x)} \left(\frac{\delta H_{\mathrm{phys}}}{\delta \partial_0 \theta(x)} \partial_0 \theta(x) + \frac{\delta H_{\mathrm{phys}}}{\delta \partial_0 \phi(x)} \partial_0 \phi(x) \right) \\ &\quad= \frac{\delta}{\delta \Pi^\sigma(x)} \left(\frac{\delta L_{\mathrm{phys}}}{\delta \partial_0 \theta(x)} \partial_0 \theta(x) + \frac{\delta L_{\mathrm{phys}}}{\delta \partial_0 \phi(x)} \partial_0 \phi(x) \right) \\ &\quad= \frac{\delta}{\delta \Pi^\sigma(x)} (B_\theta(x) \partial_0 \theta(x) + B_\phi(x) \partial_0 \phi(x)),\end{aligned} \tag{29}$$

which is a consistency condition for Eq. (28). This confirms the necessity of treating the monopole as a nondynamic field.

We now observe that

$$\frac{\delta B_\theta(x)}{\delta c^\sigma(y)} = \frac{\delta B_\phi(x)}{\delta c^\sigma(y)} = 0, \tag{30}$$

from which it follows that the EOM of c_σ receives no correction. However, its $\{,\}_{\mathrm{phys}}$ contribution corresponding to the terms in the conventional EOM for the Abelian component already contains a contribution from the monopole field strength.

Repeating the above steps for the valence gluons $\vec{X}_\mu$, assuming $\sigma \neq 0$, and combining

$$\hat{D}_0 \vec{\Pi}_\sigma(x) = \frac{\delta H}{\delta \vec{X}^\sigma(x)} - \int dy^4 \sum_{\alpha=\phi,\theta} \frac{\delta B_\alpha(y)}{\delta \vec{X}^\sigma(x)} \partial^0 \alpha(y) \tag{31}$$

with

$$\begin{aligned}\frac{\delta B_\phi(y)}{\delta \vec{X}^\sigma(x)} &= -((\sin\phi(y)_y \partial^\sigma \theta(y) \hat{n} + \hat{n}_\theta(y) \\ &\quad \times \vec{X}_\sigma(y)) \times \vec{X}_0 - \hat{n}_\phi(y) \hat{n} \cdot \vec{F}_{0\sigma}),\end{aligned} \tag{32}$$

$$\begin{aligned}\frac{\delta B_\theta(y)}{\delta \vec{X}^\sigma(x)} &= ((\partial_\sigma \phi(x) \hat{n} + \sin\phi(x) \hat{n}_\phi(x) \times \vec{X}_\sigma(x)) \\ &\quad \times \vec{X}_0 - \sin\phi \hat{n}_\theta(y) \hat{n} \cdot \vec{F}_{0\sigma}) \delta^4(x-y),\end{aligned} \tag{33}$$

gives

$$\begin{aligned}\hat{D}_0 \vec{\Pi}_\sigma(x) &= \frac{\delta H}{\delta \vec{X}^\sigma(x)} - \frac{1}{2}((\sin\phi(x)(\partial_\sigma \phi(x) \partial_0 \theta(x) \\ &\quad - \partial_\sigma \theta(x) \partial_0 \phi(x)) \hat{n}(x) \\ &\quad + (\sin\phi(x) \hat{n}_\phi(x) \partial_0 \theta(x) - \hat{n}_\theta(x) \partial_0 \phi(x)) \\ &\quad \times \vec{X}_\sigma(x)) \times \vec{X}_0 \\ &= \frac{\delta H}{\delta \vec{X}^\sigma(x)} - \frac{1}{2} g^2 (\vec{C}_\sigma(x) \times \vec{C}_0(x) + \vec{C}_0(x) \\ &\quad \times \vec{X}_\sigma(x)) \times \vec{X}_0(x).\end{aligned} \tag{34}$$

This is the converse situation of the Abelian gluon, since it is the derivative of $\vec{X}_\sigma$ that is uncorrected while its EOM receives a correction which cancels the monopole's electric contribution to $\{\hat{D}_0 \vec{X}_\sigma, H_{\mathrm{phys}}\}_{\mathrm{phys}}$. This is required by the conservation of topological current.

VI. THE FUNDAMENTAL REPRESENTATION

We consider a complex boson field $\mathbf{a}(x)$, $\mathbf{a}^\dagger(x)$ in the fundamental representation of the gauge group and probe the implications of this approach for the quark fields. Although physical quarks are fermions, we study the bosonic case to avoid distracting complications, leaving the fermionic case for a later paper.

The kinetic and interaction terms are given by

$$-(\hat{D}^\mu \mathbf{a})^\dagger(x) \hat{D}_\mu \mathbf{a}(x). \tag{35}$$

We do not consider the mass term which makes no contribution to the physics considered here.

The contribution of $\mathbf{a}(x)$, $\mathbf{a}^\dagger(x)$ to $B_\phi(x)$, $B_\theta(x)$ is

$$\begin{aligned} B_\phi(x)_{|\mathbf{a},\mathbf{a}^\dagger} &= (\hat{D}^0\mathbf{a}(x))^\dagger \hat{n}_\theta(x)\mathbf{a}(x) + (\hat{n}_\theta(x)\mathbf{a}(x))^\dagger \hat{D}^0\mathbf{a}(x), \\ B_\theta(x)_{|\mathbf{a},\mathbf{a}^\dagger} &= -(\hat{D}^0\mathbf{a}(x))^\dagger \sin\phi(x)\hat{n}_\phi(x)\mathbf{a}(x) - (\sin\phi(x)\hat{n}_\phi(x)\mathbf{a}(x))^\dagger \hat{D}^0\mathbf{a}(x), \end{aligned} \tag{36}$$

leading to a contribution of

$$\begin{aligned} \mathcal{F}_{\theta\phi}(x)_{|\mathbf{a},\mathbf{a}^\dagger} = &- (\hat{D}_0\mathbf{a}(x))^\dagger(\cos\phi(x)\hat{n}_\phi(x) - \sin\phi(x)\hat{n}(x))\mathbf{a}(x) \\ &- (\partial_0\theta(x)(\cos\phi(x)\hat{n}_\phi(x) - \sin\phi(x)\hat{n}(x))\mathbf{a}(x))^\dagger \sin\phi(x)\hat{n}_\phi(x)\mathbf{a} \\ &- ((\cos\phi(x)\hat{n}_\phi(x) - \sin\phi(x)\hat{n}(x))\mathbf{a})^\dagger \hat{D}_0\mathbf{a}(x) \\ &- (\sin\phi(x)\hat{n}_\phi(x)\mathbf{a}(x))^\dagger(\cos\phi(x)\hat{n}_\phi(x) - \sin\phi(x)\hat{n}(x))\partial_0\theta(x)\mathbf{a} \\ &+ (\hat{n}_{\theta\theta}(x)\mathbf{a}(x)\partial_0\phi(x))^\dagger \hat{n}_\theta(x)\mathbf{a}(x) \\ &+ (\hat{n}_\theta(x)\mathbf{a}(x))^\dagger \hat{n}_{\theta\theta}(x)\mathbf{a}(x)\partial_0\phi(x) \\ &- (\hat{D}_0\mathbf{a}(x))^\dagger \hat{n}_{\theta\theta}(x)\mathbf{a}(x) - (\hat{n}_{\theta\theta}(x)\mathbf{a}(x))^\dagger \hat{D}_0\mathbf{a}(x) \end{aligned} \tag{37}$$

to the q^α curvature. It follows that the complete expression for the q^α curvature in this theory is the sum of Eqs. (26) and (37).

As with the gluon d.o.f., the nonzero $\mathcal{F}_{\theta\phi}(x)$ leads to the cancellation of the monopole interactions and generates corrections to the canonical commutation relations.

VII. MONOPOLE CORRECTIONS TO THE QUANTUM COMMUTATION RELATIONS

Corrections to the classical Poisson bracket correspond to corrections to the equal-time commutators in the quantum regime. Denoting conventional commutators as $[,]_{\text{phys}}$ and the corrected ones as $[,]_{\text{new}}$, for μ, $\nu \neq 0$ we have

$$\begin{aligned} [c_\mu(x), c_\nu(z)]_{\text{new}} =& [c_\mu(x), c_\nu(z)]_{\text{phys}} - \int dy^4 \left(\frac{\delta B_\theta(y)}{\delta\Pi^\mu(x)}\mathcal{F}^{-1}_{\theta\phi}(z)\frac{\delta B_\phi(y)}{\delta\Pi^\nu(z)} - \frac{\delta B_\phi(y)}{\delta\Pi^\mu(x)}\mathcal{F}^{-1}_{\phi\theta}(z)\frac{\delta B_\theta(y)}{\delta\Pi^\nu(z)}\right)\delta^4(x-z) \\ =& [c_\mu(x), c_\nu(z)]_{\text{phys}} - \sin\phi(x)\sin\phi(z)(\partial_\mu\phi(x)\partial_\nu\theta(z) - \partial_\nu\phi(z)\partial_\mu\theta(x))\mathcal{F}^{-1}_{\theta\phi}(z)\delta^4(x-z). \end{aligned} \tag{38}$$

The second term on the final line, after integration over d^4z, clearly becomes

$$H_{\mu\nu}(x)\sin\phi(x)\mathcal{F}^{-1}_{\theta\phi}(x), \tag{39}$$

indicating the role of the monopole condensate in the correction. By contrast, the commutation relations

$$[c_\mu(x), \Pi_\nu(z)]_{\text{new}} = [c_\mu(x), \Pi_\nu(z)]_{\text{phys}}, \qquad [\Pi_\mu(x), \Pi_\nu(z)]_{\text{new}} = [\Pi_\mu(x), \Pi_\nu(z)]_{\text{phys}} \tag{40}$$

are unchanged. Nonetheless, the deviation from the canonical commutation shown in Eq. (38) is inconsistent with the particle creation/annihilation operator formalism of conventional second quantization.

Repeating for the valence gluons,

$$\begin{aligned} [\Pi^a_\mu(x), \Pi^b_\nu(z)]_{\text{new}} &= [\Pi^a_\mu(x), \Pi^b_\nu(z)]_{\text{phys}} - \int dy^4 \left(\frac{\delta B_\theta(y)}{\delta X^\mu_a(x)}\frac{\delta B_\phi(y)}{\delta X^\nu_b(z)} - \frac{\delta B_\phi(y)}{\delta X^\mu_a(x)}\frac{\delta B_\theta(y)}{\delta X^\nu_b(z)}\right)\mathcal{F}^{-1}_{\theta\phi}(z) \\ &= [\Pi^a_\mu(x), \Pi^b_\nu(z)]_{\text{phys}} + (\sin\phi(z) n^a_\phi(x) n^b_\theta(z)\vec{F}^{0\mu}(x)\cdot\hat{n}(x)\vec{F}^{0\nu}(z)\cdot\hat{n}(z) \\ &\quad - \sin\phi(x) n^a_\theta(x) n^b_\phi(z)\vec{F}^{0\mu}(z)\cdot\hat{n}(z)\vec{F}^{0\nu}(x)\cdot\hat{n}(x)) \times \mathcal{F}^{-1}_{\theta\phi}(z)\delta^4(x-z), \end{aligned} \tag{41}$$

where the second term on the final line integrates over d^4z to become

$$(n^a_\phi(x)n^b_\theta(x) - n^a_\theta(x)n^b_\phi(x))\sin\phi(x)\vec{F}^{0\mu}(x)\cdot\hat{n}(x)\vec{F}^{0\nu}(x)\cdot\hat{n}(x)\mathcal{F}^{-1}_{\theta\phi}(x), \tag{42}$$

while

$$[X^a_\mu(x), \Pi^b_\nu(z)]_{\text{new}} = [X^a_\mu(x), \Pi^b_\nu(z)]_{\text{phys}},$$
$$[X^a_\mu(x), X^b_\nu(z)]_{\text{new}} = [X^a_\mu(x), X^b_\nu(z)]_{\text{phys}}. \tag{43}$$

Indeed, this is not an exhaustive presentation of deviations from canonical quantisation. If a q^α-gauge field's derivative with respect to any physical field or its conjugate momentum is nonzero, then that field's quantization conditions and particle interpretation are affected unless the q^α curvature is exactly zero. Hence, any field interacting with the monopole component ceases to have a particle interpretation in the presence of the monopole component. In particular, its particle number becomes ill-defined, which is reminiscent of the parton model.

Equation (38) has a superficial similarity to Dirac brackets. The difference between our new brackets $\{,\}_{\text{new}}$ and Dirac brackets is clarified in Appendix B of [5]. If one introduces additional "nonphysical" momenta p_α (Eq. (B1) in [5] or Sec. 5 of [6]) corresponding to the nonphysical coordinates q_α, then the new bracket in the fully extended phase space becomes the Dirac bracket. But then we obtain constraints, especially the complicated second-stage constraint Eqs. (B5) of [6], which are absent in our approach. Equation (41) can, therefore, be considered a new shortened version of quantization for singular systems, as described in the conclusions of [5,6].

Arguments that colored states are ill-defined in the infrared regime, based on either unitarity and/or gauge invariance [24–26] date back several decades, but, to our knowledge, we are the first to argue that canonical quantization breaks down.

VIII. DISCUSSION

We have applied the Clairaut-type formalism to the CDG decomposition. This has shed light on the dynamics of the topologically generated chromomonopole field of QCD. In particular, it addresses the issue of its EOMs, or lack thereof [15,18], and the contribution its d.o.f. make to the evolution of other fields.

Indeed, the q^α curvature was found to be nonzero, leading to corrections to the time derivatives of the gluon's dynamic d.o.f., which cancel all interactions between physical and nonphysical fields from the EOMs. This is both necessary for the consistency of Eq. (28) and qualitatively consistent with our later finding that the chromomonopole background alters the canonical commutation relations in such a way as to invalidate the particle interpretation of the physical d.o.f.

This can be taken to mean that quarks and gluons do not have a well-defined particle number in the monopole condensate, suggestive of both confinement and the parton model, but it remains to repeat this work with a fully quantized, i.e., including ghosts, $SU(3)$ gauge field, and with fermionic quarks rather than scalar ones. Furthermore, while many papers have found the monopole condensate [27–29], especially with the CDG decomposition [15,20,30,31], to be energetically favorable to the perturbative vacuum, this result needs to be repeated within the Clairaut-based quantization scheme of this paper before strong claims are made.

In summary, this approach offers a rigorous analytic tool for elucidating the role of topological d.o.f. in the dynamics of quantum field theories and finds that colored states have an ill-defined particle number in the presence of nonzero monopole field strength.

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APPENDIX: THE CLAIRAUT-TYPE FORMALISM

Here we review the main ideas and formulas of the Clairaut-type formalism for singular theories [5,23]. Let us consider a singular Lagrangian $L(q^A, v^A) = L^{\deg}(q^A, v^A)$, $A = 1, \ldots, n$, which is a function of $2n$ variables (n generalized coordinates q^A and n velocities $v^A = \dot{q}^A = dq^A/dt$) on the configuration space $\mathsf{T}M$, where M is a smooth manifold, for which the Hessian's determinant is zero. Therefore, the rank of the Hessian matrix $W_{AB} = \frac{\partial^2 L(q^A, v^A)}{\partial v^B \partial v^C}$ is $r < n$, and we suppose that r is constant. We can rearrange the indices of W_{AB} in such a way that a nonsingular minor of rank r appears in the upper left corner. Then, we represent the index A as follows: if $A = 1, \ldots, r$, we replace A with i (the "regular" index), and if $A = r+1, \ldots, n$ we replace A with α (the "degenerate" index). Obviously, $\det W_{ij} \neq 0$, and $\operatorname{rank} W_{ij} = r$. Thus, any set of variables labeled by a single index splits as a disjoint union of two subsets. We call those subsets regular (having latin indices) and degenerate (having greek indices). As was shown in [5,23], the "physical" Hamiltonian can be presented in the form

$$H_{\text{phys}}(q^A, p_i) = \sum_{i=1}^{r} p_i V^i(q^A, p_i, v^\alpha) + \sum_{\alpha=r+1}^{n} B_\alpha(q^A, p_i) v^\alpha - L(q^A, V^i(q^A, p_i, v^\alpha), v^\alpha), \tag{A1}$$

where the functions

$$B_\alpha(q^A, p_i) \stackrel{\text{def}}{=} \left.\frac{\partial L(q^A, v^A)}{\partial v^\alpha}\right|_{v^i = V^i(q^A, p_i, v^\alpha)} \tag{A2}$$

are independent of the unresolved velocities v^α since rank $W_{AB} = r$. Also, the rhs of (A1) does not depend on the degenerate velocities v^α,

$$\frac{\partial H_{\text{phys}}}{\partial v^\alpha} = 0, \tag{A3}$$

which justifies the term physical. The Hamilton-Clairaut system, which describes any singular Lagrangian classical system (satisfying the second order Lagrange equations), has the form

$$\frac{dq^i}{dt} = \{q^i, H_{\text{phys}}\}_{\text{phys}} - \sum_{\beta=r+1}^{n} \{q^i, B_\beta\}_{\text{phys}} \frac{dq^\beta}{dt}, \quad i = 1, \ldots, r, \tag{A4}$$

$$\frac{dp_i}{dt} = \{p_i, H_{\text{phys}}\}_{\text{phys}} - \sum_{\beta=r+1}^{n} \{p_i, B_\beta\}_{\text{phys}} \frac{dq^\beta}{dt}, \quad i = 1, \ldots, r, \tag{A5}$$

$$\sum_{\beta=r+1}^{n} \left[\frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \{B_\alpha, B_\beta\}_{\text{phys}}\right] \frac{dq^\beta}{dt} = \frac{\partial H_{\text{phys}}}{\partial q^\alpha} + \{B_\alpha, H_{\text{phys}}\}_{\text{phys}}, \quad \alpha = r+1, \ldots, n, \tag{A6}$$

where the physical Poisson bracket (in regular variables q^i, p_i) is

$$\{X, Y\}_{\text{phys}} = \sum_{i=1}^{n-r} \left(\frac{\partial X}{\partial q^i}\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial q^i}\frac{\partial X}{\partial p_i}\right). \tag{A7}$$

Whether the variables $B_\alpha(q^A, p_i)$ have a nontrivial effect on the time evolution and commutation relations is equivalent to whether or not the so-called "q^α-field strength"

$$\mathcal{F}_{\alpha\beta} = \frac{\partial B_\beta}{\partial q^\alpha} - \frac{\partial B_\alpha}{\partial q^\beta} + \{B_\alpha, B_\beta\}_{\text{phys}} \tag{A8}$$

is nonzero. See [5,6,23] for more details.

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